

# Mutual friction in superfluid neutron stars

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5 February 2008

## ABSTRACT

We discuss vortex-mediated mutual friction in the two-fluid model for superfluid neutron star cores. Our discussion is based on the general formalism developed by Carter and collaborators, which makes due distinction between transport velocity and momentum for each fluid. This is essential for an implementation of the so-called entrainment effect, whereby the flow of one fluid imparts momentum in the other and vice versa. The mutual friction follows by balancing the Magnus force that acts on the quantised neutron vortices with a resistive force due to the scattering of electrons off of the magnetic field with which each vortex core is endowed. We derive the form of the macroscopic mutual friction force which is relevant for a model based on smooth-averaging over a collection of vortices. We discuss the coefficients that enter the expression for this force, and the timescale on which the two interpenetrating fluids in a neutron star core are coupled. This discussion confirms that our new formulation accords well with previous work in this area.

## 1 INTRODUCTION

A superfluid rotates by forming a dense array of quantised vortices. In the case of mature neutron stars, which are expected to exhibit large scale superfluidity since their core temperatures are orders of magnitude below the Fermi temperatures for both neutrons and protons, these vortices should play a key role in determining the rotational dynamics. In fact, the sudden spin-up associated with the observed radio pulsar glitches (Lyne, Shemar & Graham Smith 2000), and the relaxation that follows, is commonly viewed as strong evidence of transfer of angular momentum between a superfluid component and the charged component to which the star's magnetic field is locked and which, presumably, is linked to the pulsar emission mechanism. Hence, the notion that a neutron star acts as a kilometer-sized superfluid system is strongly supported, both theoretically and observationally.

The purpose of this study is to model vortex-mediated dissipation in the two-fluid paradigm for superfluid neutron stars. The common view is that the most important dissipation mechanism in a superfluid neutron star core originates from electrons scattering off of the magnetic fields associated with the individual vortex cores (Alpar, Langer & Sauls 1984; Mendell 1991b). Key to this idea is, as discussed by Sauls, Stein & Serene (1982) (see also Vardanyan & Sedrakyan (1981)), the fact that the entrainment effect induces a flow in the proton fluid around each neutron vortex. This, in turn, generates a local magnetic field of the order of  $10^{14}$  G off of which the electrons scatter dissipatively (Alpar, Langer & Sauls 1984). The outcome is a coupling between the neutrons and the interpenetrating conglomerate of charged particles. This mechanism has been considered in two important scenarios. First, Alpar & Sauls (1988) have argued that it leads to the core fluids coupling on a timescale of  $400 - 10^4$  rotation periods. (For alternative pictures, where the coupling timescale is significantly different, see Sedrakyan, Shakhbasyan & Movsisyan (1985); Sedrakian & Sedrakian (1997); Sedrakian (1998).) As this is far faster than the relaxation timescale following a Vela pulsar glitch, one can argue that the glitches cannot be associated with the core but rely on the conditions in the crust. Second, Mendell (1991b) discussed the fact that the mutual friction is also an important damping agent for neutron star oscillations. This is of key importance for potential gravitational-wave driven mode-instabilities (see Andersson (2003) for a recent review). In fact, the current thinking is that the mutual friction suppresses the instability in the star's f-mode entirely (Lindblom & Mendell 1995). The effect on the unstable r-modes is not quite as devastating (Lindblom & Mendell 2000), but it could well be that the mutual friction sets the most stringent constraints on the instability window also in this case.

In recent work we have applied the general formalism for superfluid neutron stars in Newtonian gravity developed by Prix

(2004) (see also the closely related work by Carter & Chamel (2004, 2005a,b)) to a set of problems relevant for astrophysical neutron stars. In particular, we have considered rotating stellar configurations where the two fluids are allowed to spin at different rates (Prix, Comer & Andersson 2002; Andersson & Comer 2001; Prix, Novak & Comer 2005). We have also discussed the nature of the inertial modes of oscillation (of which the r-modes form a sub-class) (Prix, Comer & Andersson 2004), and investigated the possibility that a two-stream instability may operate in a superfluid neutron star (Andersson, Comer & Prix 2003, 2004). These papers demonstrate clearly i) that the superfluid oscillation problem is richer than tends to be assumed, and ii) that the entrainment effect (whereby the flow of one fluid imparts momentum in the other) plays a key role in determining the nature of the various modes of oscillation and the extent to which the various fluids partake in the pulsation.

Our previous studies were based on the equations that follow after smooth-averaging over a collection of vortices. This “macroscopic” approach does not provide insight into the actual vortex dynamics. In order to devise a model for the required mutual friction force we must make connection between the large scale dynamics that we have previously considered and the “mesoscopic” level, which is sufficiently resolved that individual vortices can be distinguished yet sufficiently coarse that we do not have to worry about “microscopic” quantum effects (other than the quantisation of vorticity). Our analysis is based on the well-established procedure for deducing the mutual friction force in the case of superfluid Helium (Hall & Vinen 1956; Bekarevich & Khalatnikov 1961; Donnelly 1991; Barenghi, Donnelly & Vinen 2001), and proceeds in three main steps: First we discuss the nature of the quantised vorticity, making the appropriate distinction between transport velocities and momenta. This is important if one wants to correctly account for the entrainment. Next we derive an expression for the Magnus force, which describes how a bulk flow imparts a force on the vortices (analogous to the so-called Joukowski lift in standard fluid mechanics); see Sonin (1987) for a useful review. Finally, we derive an expression for the vortex-mediated mutual friction force, which couples the superfluid neutrons to the conglomerate of charged components. Having derived this expression we discuss the relevant coefficients and compare our final results to previous ones in the literature.

## 2 THE SUPERFLUID EQUATIONS OF MOTION

We take as our starting point the two-fluid equations derived by, for instance, Prix (2004) (see also Andersson & Comer (2005)). In this description the number density of each fluid obeys the continuity equation

$$\frac{\partial n_x}{\partial t} + \nabla_j (n_x v_x^j) = 0 . \quad (1)$$

Here, we distinguish between the “constituent index”  $x$  which (in the present context) can be either  $n$  or  $p$ , and the spatial index  $j$ . The index  $n$  represents the superfluid neutrons while  $p$  corresponds to a conglomerate of all charged particles (protons and electrons), which are expected to flow together due to electromagnetic coupling (Mendell 1991a). In the following, repeated constituent indices ( $x$  and  $y$ ) never imply summation while spatial indices  $i, j$  and  $k$  satisfy the Einstein summation convention. In Eq. (1),  $n_x$  is the number density and  $v_x^i$  is the transport velocity. That is,  $n_x^i = n_x v_x^i$  represents the true number density current for species  $x$ .

Each fluid satisfies an Euler-type equation, which ensures the conservation of total momentum. For constituent  $x$  this equation can be written

$$\left( \frac{\partial}{\partial t} + v_x^j \nabla_j \right) [v_i^x + \varepsilon_x w_i^{yx}] + \nabla_i (\Phi + \tilde{\mu}_x) + \varepsilon_x w_j^{yx} \nabla_i v_x^j = 0 . \quad (2)$$

Here we have defined the relative velocity

$$w_i^{yx} = v_i^y - v_i^x . \quad (3)$$

Furthermore,

$$\tilde{\mu}_x = \frac{\mu_x}{m_x} = \frac{1}{m_x} \frac{\partial E}{\partial n_x} , \quad (4)$$

where  $E$  is the internal energy of the system, is the relevant chemical potential per unit mass. The entrainment is included via the coefficients

$$\varepsilon_x = 2\rho_x \alpha \quad \text{where} \quad \alpha = \frac{\partial E}{\partial w_{np}^2} , \quad (5)$$

and  $\Phi$  represents the gravitational potential. For a detailed discussion of these equations, see Prix (2004); Andersson & Comer (2005).

Using the fact that the momentum per particle (which is canonically conjugate to the number current  $n_x^i$ ) of each fluid is defined as

$$p_i^x = m_x [v_i^x + \varepsilon_x w_i^{yx}] , \quad (6)$$

and identifying the spatial derivatives in (2) as components of the Lie derivative  $\mathcal{L}_{v_x}$  associated with the velocity  $v_x^i$ , i.e. using

$$\mathcal{L}_{v_x} W_i = v_x^j \nabla_j W_i + W_j \nabla_i v_x^j \quad (7)$$

which holds for a general co-vector  $W_i$ , we can rewrite (2) as

$$\left( \frac{\partial}{\partial t} + \mathcal{L}_{v_x} \right) p_i^x + m_x \nabla_i \left( \Phi + \tilde{\mu}_x - \frac{1}{2} v_x^2 \right) = 0 . \quad (8)$$

Let us now make contact with microphysics. In the case of a neutron superfluid the momentum  $p_i^n$  is related to the gradient of the phase  $\chi_n$  of the condensate wavefunction (the “order parameter”) via

$$p_i^n = \frac{\hbar}{2} \nabla_i \chi_n , \quad (9)$$

where the factor of 2 is introduced since we are dealing with neutron Cooper pairs. Taking the curl of this relation, we see that the superfluid is generally irrotational. It follows from (8) that this property is conserved by the flow, see Prix (2004), and hence it is natural that vortices are associated with quantised momentum circulation. To some extent, this contrasts with the “orthodox” Landau formulation of superfluids (see the work of, for example, Mendell (1991a,b)). In that paradigm  $p_i^n/m_n$  is referred to as the “superfluid velocity”,  $V_i^s$  (say). Conceptually, this is somewhat confused but there is no real risk of making significant mistakes as long as one does not try to account for entrainment. After all, if  $\varepsilon_n = 0$  we trivially have  $V_i^s = v_i^n$ . As we will see, the situation when entrainment is considered is far from trivial and one must take some care in order to avoid inconsistencies.

If we introduce vortices in the superfluid then it is easy to show that the circulation in the neutron fluid must be quantised. Integrating along a contour which encloses a single vortex we have

$$\mathcal{C} = \oint p_i^n dx^i = \int (\epsilon^{ijk} \nabla_j p_k^n) dS_i = \frac{h}{2} . \quad (10)$$

If for simplicity we assume that the vortex is straight, and introduce cylindrical coordinates  $(r, \theta, z)$  centered on the vortex, the neutron momentum can be represented by

$$p_\theta^n = \frac{\mathcal{C}}{2\pi} . \quad (11)$$

It should be noted that, in an orthonormal basis, this corresponds to the more familiar looking vortex solution  $\vec{p} = \mathcal{C} \hat{e}_\theta / 2\pi r$ . Formally, the vorticity of the flow is associated with the singularity at  $r = 0$ . However, on a macroscopic scale we can meaningfully introduce a smooth averaged “rotation velocity” by comparing the standard result

$$2\Omega_x^i = \epsilon^{ijk} \nabla_j v_k^x , \quad (12)$$

to (6). Hence, on the macroscopic level we identify (assuming that the lengthscale considered is sufficiently small that we can treat  $\varepsilon_n$  as a constant)

$$\epsilon^{ijk} \nabla_j p_k^n = 2m_n [\Omega_n^i + \varepsilon_n (\Omega_p^i - \Omega_n^i)] . \quad (13)$$

As a result, we find that by enclosing  $\mathcal{N}$  vortices we get

$$2\pi \int_0^r m_n [\Omega_n + \varepsilon_n (\Omega_p - \Omega_n)] r dr = \frac{\mathcal{N}h}{2} , \quad (14)$$

where we have used  $\Omega_x^i = \Omega_x \hat{e}_z^i$ . Defining the surface vortex density  $n_v$  as

$$n_v = \frac{d\mathcal{N}}{2\pi r dr} , \quad (15)$$

we see that

$$n_v \kappa = 2[\Omega_n + \varepsilon_n (\Omega_p - \Omega_n)] . \quad (16)$$

It should be noted that this result, which clearly displays the interpenetrating nature of the two fluids, differs from the “standard” result (see for example Alpar, Langer & Sauls (1984)). Because of the entrainment effect, the number density of neutron vortices depends explicitly on the rotation of the proton fluid (this point was recently discussed also by Chamel & Carter (2005)).

Let us now consider the force acting on a single neutron vortex. This means that the neutron momentum is represented by (11), since we focus on the mesoscopic scale. Meanwhile, there should (on this scale) be no circulation in the proton momentum. We will motivate this assumption later when we account for the fact that the protons are charged. Taking  $p_{vp}^i = 0$  we have

$$\frac{p_\theta^{vn}}{m_n} = v_\theta^{vn} + \varepsilon_n (v_\theta^{vp} - v_\theta^{vn}) = \frac{\kappa}{2\pi} , \quad (17)$$

$$\frac{p_\theta^{vp}}{m_p} = v_\theta^{vp} + \varepsilon_p (v_\theta^{vn} - v_\theta^{vp}) = 0 , \quad (18)$$

where  $\kappa = h/2m_n$ . For later convenience we have identified the momentum associated with the vortex flow  $p_{v_x}^i$  by an additional index  $v$ . This will later allow us to distinguish between a uniform flow past, and the rotation induced by, the vortex. If we solve for the individual flows induced by the vortex we find

$$v_{\theta}^{vn} = \left( \frac{\varepsilon_p - 1}{\varepsilon_p + \varepsilon_n - 1} \right) \frac{\kappa}{2\pi} , \quad (19)$$

$$v_{\theta}^{vp} = \left( \frac{\varepsilon_p}{\varepsilon_p + \varepsilon_n - 1} \right) \frac{\kappa}{2\pi} . \quad (20)$$

### 3 THE MAGNUS FORCE

Having discussed the nature of the quantised vorticity, we are ready to investigate the force acting on a vortex due to a uniform flow past it. To do this, we require the flow of momentum  $\pi_x^i = n_x p_x^i$ . That is, we need

$$\frac{\partial \pi_x^i}{\partial t} = \frac{\partial}{\partial t}(n_x p_x^i) = n_x \frac{\partial p_x^i}{\partial t} + p_x^i \frac{\partial n_x}{\partial t} . \quad (21)$$

Combining (1) and (2) we can show that

$$\begin{aligned} \frac{\partial}{\partial t}(\pi_i^n + \pi_i^p) &= -\rho \nabla_i \Phi - n_n \nabla_i \mu_n - n_p \nabla_i \mu_p + \alpha \nabla_i w_{pn}^2 - \nabla_j [n_n p_i^n v_n^j + n_p p_i^p v_p^j] \\ &\approx -\nabla_j \{ n_n p_i^n v_n^j + n_p p_i^p v_p^j + \delta_i^j [\rho \Phi + n_n \mu_n + n_p \mu_p - \alpha w_{pn}^2] \} \\ &= -\nabla_j \Pi_i^j , \end{aligned} \quad (22)$$

where  $\rho = \rho_n + \rho_p = m_n(n_n + n_p)$  is the total mass density — we take  $m_n = m_p$  throughout this paper. It has been assumed that we are working on a sufficiently small scale that we can take the number densities  $n_x$  and the entrainment parameter  $\alpha$  constant.

In order to quantify the force acting on a vortex it is natural to work in a frame in which the vortex is at rest. In that frame we want the background flow to be stationary and irrotational. Then we want to ask what the effects of introducing a single vortex in this flow may be. Translating (2) into a frame moving with velocity  $v_i^L$  we get

$$\left( \frac{\partial}{\partial t} + w_{xL}^j \nabla_j \right) [w_i^{xL} + \varepsilon_x w_i^{yx}] + \nabla_i (\Phi + \tilde{\mu}_x) + \varepsilon_x w_j^{yx} \nabla_i w_{xL}^j = 0 , \quad (23)$$

where  $w_i^{xL} = v_i^x - v_i^L$ . Imposing the condition of a stationary and irrotational flow, the latter of which leads to

$$w_{xL}^j \nabla_i [w_j^{xL} + \varepsilon_x w_j^{yx}] = w_{xL}^j \nabla_j [w_i^{xL} + \varepsilon_x w_i^{yx}] , \quad (24)$$

we have

$$\nabla_i \left\{ \frac{1}{2} w_{xL}^2 + \Phi + \tilde{\mu}_x + \varepsilon_x w_{xL}^j w_j^{xy} \right\} = 0 . \quad (25)$$

After integration, this yields

$$\frac{1}{2} w_{xL}^2 + \Phi + \tilde{\mu}_x + \varepsilon_x w_{xL}^j w_j^{xy} = C_x = \text{constant} . \quad (26)$$

From these two equations, we have

$$D = \rho_n C_n + \rho_p C_p = \frac{1}{2} \rho_n w_{nL}^2 + \frac{1}{2} \rho_p w_{pL}^2 + \rho \Phi + n_n \mu_n + n_p \mu_p - 2\alpha w_{pn}^2 . \quad (27)$$

Combining this with (22), obviously translated into the vortex frame, we can show that

$$\Pi_i^j = \rho_n w_{nL}^j w_i^{nL} + \rho_p w_{pL}^j w_i^{pL} - 2\alpha w_{pn}^j w_i^{pn} + \delta_i^j \left[ D - \frac{1}{2} \rho_n w_{nL}^2 - \frac{1}{2} \rho_p w_{pL}^2 + \alpha w_{pn}^2 \right] . \quad (28)$$

The force per unit length that acts on the vortex is now determined by the flow of momentum through a cylinder enclosing the vortex. That is, we need to evaluate

$$f_i = \oint_C \Pi_i^j s_j dl , \quad (29)$$

where  $C$  encloses the vortex and  $s_j$  is the unit normal to the cylinder. For a single vortex we can assume the flow to be approximately of the form

$$w_i^{xL} = U_i^x + v_i^{vx} , \quad (30)$$

where  $U_i^x$  is uniform, stationary and irrotational in the vortex frame. The vortex flows  $v_i^{vx}$  are given by (19) and (20). Substituting in (28) and retaining only those terms that will eventually contribute to the integral in (29) we find

$$\Pi_i^j s_j = n_n U_n^j (p_i^{vn} s_j - p_j^{vn} s_i) + n_p U_p^j (p_i^{vp} s_j - p_j^{vp} s_i) . \quad (31)$$

To arrive at this result we have used the fact that all coefficients of  $s_i$  which are constant on the contour vanish when integrated around a circle. As the vector  $s_i$  in polar coordinates is proportional to the radial vector and  $v_{vx}^i$  is in the  $\theta$  direction we obviously have  $v_{vx}^j s_j = 0$ . Finally, as  $U_x^i$  is a constant flow, the integral around the vortex of  $(U_x^j s_j) U_i^x$  vanishes. Our final expression can be rewritten as

$$\Pi_i^j s_j = n_n \epsilon_{ijk} U_n^j (\epsilon^{klm} p_l^{vn} s_m) , \quad (32)$$

where we have used the fact that  $p_i^{vp} = 0$  for a single neutron vortex.

If we define the “vorticity”  $\kappa^i$  as a vector with magnitude  $\kappa$  which is aligned with  $\epsilon^{ijk} \nabla_j p_k^n$ , i.e. use

$$\epsilon_{ijk} p_{vn}^j s^k = -\frac{m_n \kappa_i}{2\pi} , \quad (33)$$

(in orthonormal cylindrical coordinates  $\vec{\kappa} = \kappa \hat{e}_z$ ), and work out the force integral, we arrive at the final result for the Magnus force acting on the vortex

$$f_i^M = \rho_n \epsilon_{ijk} U_n^j \kappa^k = -\rho_n \epsilon_{ijk} w_{nL}^j \kappa^k = \rho_n \epsilon_{ijk} \kappa^j w_{nL}^k . \quad (34)$$

From this result, the force density (per unit volume) on a collection of vortices follows readily as  $n_v f_i^M$ .

It is important to note that entrainment affects the Magnus force in two ways. First of all, it enters (34) via  $\kappa^j$ . Secondly, it also impacts on the vortex number density  $n_v$  according to (16). These results agree with the discussion of Chamel & Carter (2005). Our final formula (34) also agrees with the result used by Langlois, Sedrakian & Carter (1998). Mendell (1991b) makes use of a more generic expression which allows for the presence of vortices in each different fluid. The coupling coefficients in that expression are, however, left unspecified.

#### 4 THE MUTUAL FRICTION FORCE

Having found the form of the Magnus force, we can determine the “mutual friction” force, which represents a balance between the Magnus force and standard “resistivity” due to electrons scattering off the magnetic field associated with each vortex. Taking the latter force to be proportional to the difference in velocity between the vortex and the charged fluid flow (protons and electrons), we have

$$f_i^e = \mathcal{R} (v_i^p - v_i^L) . \quad (35)$$

Assuming that the vortex can be treated as massless (see Mendell (1991a) for a justification of this assumption), this force must equal the Magnus force given by (34). Solving the resultant equation for  $v_i^L$  (using repeated cross products with  $\kappa^i$ , see for example Hall & Vinen (1956)) we find

$$\begin{aligned} v_L^i &= v_p^i + \frac{\mathcal{R}}{\rho_n \kappa^2} \left( \frac{1}{1 + \mathcal{R}^2 / \rho_n^2 \kappa^2} \right) \epsilon^{ijk} \kappa_j w_k^{pn} \\ &\quad + \frac{1}{\kappa^2} \left( \frac{1}{1 + \mathcal{R}^2 / \rho_n^2 \kappa^2} \right) \epsilon^{ijk} \kappa_j \epsilon_{klm} \kappa^l w_{pn}^m . \end{aligned} \quad (36)$$

Consequently, the force per unit length acting on the vortex is

$$\begin{aligned} f_e^i &= \mathcal{R} (v_p^i - v_L^i) \\ &= \frac{\mathcal{R}^2}{\rho_n \kappa^2} \left( \frac{1}{1 + \mathcal{R}^2 / \rho_n^2 \kappa^2} \right) \epsilon^{ijk} \kappa_j w_k^{pn} \\ &\quad + \frac{\mathcal{R}}{\kappa^2} \left( \frac{1}{1 + \mathcal{R}^2 / \rho_n^2 \kappa^2} \right) \epsilon^{ijk} \kappa_j \epsilon_{klm} \kappa^l w_{pn}^m . \end{aligned} \quad (37)$$

The first term in this expression is analogous to the Magnus force, although now expressed in terms of the velocity difference  $w_k^{pn}$ . As this force is perpendicular to the relative velocity, it is non-dissipative. The second term, on the other hand, can be rewritten using

$$\epsilon^{ijk} \kappa_j \epsilon_{klm} \kappa^l w_{pn}^m = \kappa^2 (\hat{\kappa}^i \hat{\kappa}^j - g^{ij}) w_j^{pn} , \quad (38)$$

where the bracket can be recognised as the projection orthogonal to  $\hat{\kappa}^i = \kappa^i / \kappa$ . This term induces dissipation in the flow.

From (36) and (37) we also see that:

- In the limit  $\mathcal{R} \rightarrow \infty$  we must have  $v_i^L \rightarrow v_i^p$ . That is, the neutron vortices are strongly coupled to the charged fluid.
- In the opposite, “weak coupling”, limit where  $\mathcal{R} \rightarrow 0$ , the vortices must flow with the neutron fluid, i.e., we have  $v_i^L = v_i^n$ .
- The dissipative part of the mutual friction force, somewhat counterintuitively, vanishes in both of these limits.

The form of the mutual friction to be used in the equations of motion is obtained, assuming that there is no direct vortex interaction (see Ruderman, Zhu & Chen (1998) for a discussion of such interactions), by multiplying (37) by the vortex density  $n_v$ . This is then the force that acts on the neutron superfluid, eg. which enters the right-hand side of Eq. (2) with  $x = n$ . An equal and opposite force acts on the charged conglomerate, and provides the right-hand side of Eq. (2) with  $x = p$ .

## 5 ESTIMATING THE COEFFICIENTS

To complete our investigation, and in order to facilitate the use of our results in studies of the dynamics of neutron stars, we need to discuss the parameters that determine the strength of the mutual friction force. In essence, we need to estimate the “friction coefficient”  $\mathcal{R}$ . To do this, we rely on the previous work of Alpar, Langer & Sauls (1984) and Mendell (1991b). Below we “translate” their analysis into our formalism.

The contribution to the mutual friction force which is expected to provide the dominant coupling mechanism between the superfluid neutrons and the charged conglomerate is due to electrons scattering off the magnetic field associated with each vortex. An early analysis of this coupling was carried out by Sauls, Stein & Serene (1982), who discussed the importance of the spontaneous magnetisation of the vortex. Shortly after this analysis it was realised that the proton current induced by the entrainment effect would lead to a significantly stronger magnetic field (Alpar, Langer & Sauls 1984). Hence, we focus our attention on this case.

For superconducting protons, the expression for the momentum must be replaced by (see, for example, Prix (2005))

$$p_i^p + \frac{e}{c} A_i = \frac{\hbar}{2} \nabla_i \chi_p . \quad (39)$$

(Here and in the following we are using Gaussian units). The magnetic field associated with the flow follows from, firstly the definition of the magnetic potential

$$B_i = \epsilon_{ijk} \nabla^j A^k \quad (40)$$

and secondly, the Maxwell equation

$$j_i = \frac{c}{4\pi} \epsilon_{ijk} \nabla^j B^k = e n_p v_i^p , \quad (41)$$

where the right-hand side is the charge current. In writing down this relation we have adopted the convention that the charge currents affect only the magnetic induction  $B_i$ , see Tilley & Tilley (1990) for further discussion. It is now straightforward to combine (40) and (41) to obtain the London equation for  $A_i$ .

Presently, we are primarily interested in the magnetic field generated by the flow of entrained protons around a single neutron vortex. This means that it is natural to assume that there are no fluxtubes in the proton fluid. This is equivalent to assuming that the phase  $\chi_p$  is smooth. Then a gauge transformation can be made such that (39) is replaced by, see Tilley & Tilley (1990),

$$p_i^p + \frac{e}{c} A_i = 0 . \quad (42)$$

Thus it follows that

$$m_p(1 - \varepsilon_p)v_i^p = -\frac{e}{c} A_i - m_p \varepsilon_p v_i^n , \quad (43)$$

and we arrive at the following equation for the magnetic flux (using the fact that  $\nabla_i B^i = 0$  and assuming that the vortex is represented by a delta-function)

$$\nabla^2 B_i - \frac{1}{\Lambda_*^2} B_i = \frac{8\pi e \alpha}{m_p m_n c} \left(1 - \frac{2\alpha\rho}{\rho_n \rho_p}\right)^{-1} \epsilon_{ijk} \nabla^j p_n^k . \quad (44)$$

Here, the effective London penetration length  $\Lambda_*$  is defined by

$$\Lambda_*^2 = \frac{c^2 m_p^2}{4\pi e^2 \rho_p} \left(1 - \frac{2\alpha\rho}{\rho_n \rho_p}\right) \left(1 - \frac{2\alpha}{\rho_n}\right)^{-1} . \quad (45)$$

Comparing (44) to equation (14) of Alpar, Langer & Sauls (1984), and recalling the fact that the “superfluid velocity” in the orthodox formalism is in fact the rescaled momentum, we find that we should identify

$$\rho_s^{pp} = \rho_p \left(1 - \frac{2\alpha\rho}{\rho_n \rho_p}\right)^{-1} \left(1 - \frac{2\alpha}{\rho_n}\right) , \quad \text{and} \quad \rho_s^{pn} = -2\alpha \left(1 - \frac{2\alpha\rho}{\rho_n \rho_p}\right)^{-1} . \quad (46)$$

The relationship between the two formalisms has already been discussed by Prix, Comer & Andersson (2002). Using their analysis we readily demonstrate that the above identification is correct.

As the equations for the magnetic field are identical, the required solution is identical to that given in previous work

(Alpar, Langer & Sauls 1984; Mendell 1991a). Solving (44) for the case where the vortex is represented by a delta-function (Fetter & Hohenberg 1969), we readily find that the only non-vanishing component of the magnetic field is

$$B^z = \frac{\Phi_*}{2\pi\Lambda_*^2} K_0(r/\Lambda_*) . \quad (47)$$

Here  $K_0$  is a modified Bessel function,

$$\Phi_* = \frac{hc}{2e} \frac{m_p}{m_n} \frac{\rho_s^{\text{pn}}}{\rho_s^{\text{pp}}} , \quad (48)$$

and  $m_p = m_n$  for all practical purposes. Crucially, this means that

$$|\vec{A}| \propto K_1(r/\Lambda_*) \sim \sqrt{\frac{\Lambda_*}{r}} \exp(-r/\Lambda_*) \quad \text{for } r \gg \Lambda_* \quad (49)$$

This resolves the apparent contradiction between (18), which formed a key part of our derivation of the Magnus force, and the correct formula for charged protons, Eq. (42). We see that the analysis in Section III holds, provided that the force calculation can be performed sufficiently far away from the vortex that  $A_i$  can be neglected yet close enough that the assumption of essentially constant densities, entrainment parameters etcetera holds. This should always be possible, since  $\Lambda_*$  is many orders of magnitude smaller than eg. the intervortex separation.

For later convenience, it is useful to take a brief detour at this point and introduce the effective proton mass  $m_p^*$ . This concept was also discussed by Prix, Comer & Andersson (2002). It is natural that the entrainment effect can be expressed in terms of an altered “effective” mass since it couples the momenta of the two fluids. The analysis leading to this is, in fact, very simple. In a frame comoving with the neutrons, i.e. in which  $v_i^n = 0$ , (and well away from all vortices in a region where we can neglect the magnetic field, according to the discussion above) we have

$$p_i^p = m_p(1 - \varepsilon_p)v_i^p \equiv m_p^*v_i^p . \quad (50)$$

Hence, it follows that

$$2\alpha = \rho_p \varepsilon_p = n_p(m_p - m_p^*) = n_p \delta m_p^* . \quad (51)$$

(Note that we could alternatively have defined the effective mass in the frame where  $p_i^n = 0$ . As discussed by Prix, Comer & Andersson (2002), the result is the same provided that the proton fraction is small — the limit which is relevant for neutron star cores.)

This means that we can use

$$\rho_s^{\text{np}} = -\rho_p \left( \frac{\delta m_p^*}{m_p} \right) \left( 1 - \frac{\rho}{\rho_n} \frac{\delta m_p^*}{m_p} \right)^{-1} \approx -\rho_p \left( \frac{\delta m_p^*}{m_p^*} \right) , \quad (52)$$

which should be accurate in the case of neutron stars where the proton fraction  $x_p = \rho_p/\rho$  is small. We also have

$$\rho_s^{\text{pp}} = \rho_p \frac{\rho_n m_p - \rho_p \delta m_p^*}{\rho_n m_p - \rho \delta m_p^*} \approx \rho_p \left( \frac{m_p}{m_p^*} \right) . \quad (53)$$

These expressions are, not surprisingly, identical to those given by Alpar, Langer & Sauls (1984). Given these approximations, the penetration length is

$$\Lambda_* \approx 1.3 \times 10^2 \left[ \left( \frac{x_p}{0.05} \right) \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right) \frac{m_p}{m_p^*} \right]^{-1/2} \text{ fm} , \quad (54)$$

and the magnetic field associated with the vortex core is approximated by (Alpar, Langer & Sauls 1984)

$$B \approx \frac{|\Phi_*|}{2\pi\Lambda_*^2} \approx 1.9 \times 10^{14} \text{ G} \left( \frac{x_p}{0.05} \right) \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right) \left| \frac{\delta m_p^*}{m_p^*} \right| . \quad (55)$$

In order to estimate the relaxation time for electrons scattered off the vortex magnetic fields, we combine three further results from Alpar, Langer & Sauls (1984). The first is the relaxation timescale  $\tau_0$  in the limit of a vanishing vortex radius. It follows as

$$\tau_0^{-1} = \pi N_\tau \Phi_*^2 , \quad (56)$$

where

$$N_\tau = \frac{2\pi}{\hbar} n_v \left( \frac{e\hbar}{2m_e c} \right)^2 \left( \frac{m_e c^2}{E_{Fe}} \right)^2 \frac{E_{Fe}}{(\pi\hbar c)^2} . \quad (57)$$

We will assume that the electrons are ultrarelativistic, i.e. use

$$E_{Fe} = \hbar c k_{Fe}, \quad \text{where} \quad k_{Fe} = (3\pi^2 n_e)^{1/3} = (3\pi^2 n_p)^{1/3}. \quad (58)$$

As the electrons and protons are expected to couple on a much shorter timescale we account for the increased inertia by using

$$\tau_v = \left( \frac{m_p c^2}{\hbar c k_{Fe}} \right) \tau_0. \quad (59)$$

The final factor encodes the dependence on the finite size of the scattering centre. As discussed by Alpar, Langer & Sauls (1984), this leads to

$$\tau_v \longrightarrow \frac{16}{3\pi} \frac{\alpha}{\beta} \tau_v, \quad (60)$$

where

$$\frac{\alpha}{\beta} = 2k_{Fe}\Lambda_* \approx 120 \left( \frac{m_p^*}{m_p} \right)^{1/2} \left[ \left( \frac{x_p}{0.05} \right) \left( \frac{\rho_p}{10^{14} \text{g/cm}^3} \right) \right]^{-1/6}. \quad (61)$$

Having arrived at an estimate of the timescale on which the vortices relax to the motion of the charged components, we can make connection with our expression for the mutual friction force from the previous section. To do this we note that the relative velocity between vortices and charged components relaxes according to  $\partial_t \Delta v_i = -\Delta v_i / \tau_v$ , from which we can deduce that the average force (per unit length) acting on a typical vortex is

$$\langle f_i \rangle = \frac{\rho_p}{n_v \tau_v} (v_i^p - v_i^L). \quad (62)$$

Comparing this to (35) we see that

$$\mathcal{R} = \frac{\rho_p}{n_v \tau_v}. \quad (63)$$

As discussed in the previous section, it is useful to establish whether we are in the regime of strong or weak coupling. The above analysis leads to the estimate

$$\left( \frac{\mathcal{R}}{\rho_n \kappa} \right)^2 \approx 1.6 \times 10^{-7} \left( \frac{\delta m_p^*}{m_p} \right)^4 \left( \frac{m_p}{m_p^*} \right) \left( \frac{x_p}{0.05} \right)^{7/3} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/3} \ll 1. \quad (64)$$

Given that the effective proton mass is such that  $m_p^*/m_p \approx 0.5 - 0.7$  we are firmly in the weak coupling regime.

Finally, let us return to (37) — the force per unit length acting on an individual vortex — and replace it with the relevant force acting on the superfluid neutrons after averaging over the vortices. This is a slightly subtle issue. First we need to appreciate that the vortices are already accounted for in the averaged equations, eg. (8). In a sense this means that the Magnus force (34) is already contained in this equation. Hence we need to add only the resistive part (35) to the description. The force that we require thus follows simply by multiplying (37) by the local surface density of vortices  $n_v$ . Provided that we are dealing with the weak coupling limit, the resultant force acting on the neutron fluid can be written

$$f_{mf}^i = \mathcal{B} \rho_n n_v \epsilon^{ijk} \hat{\kappa}_j \epsilon_{klm} \kappa^l w_{pn}^m + \mathcal{B}' \rho_n n_v \epsilon^{ijk} \kappa_j w_k^{pn}. \quad (65)$$

An equal and opposite force acts on the proton fluid. Here

$$\mathcal{B} = \frac{\mathcal{R}}{\rho_n \kappa} \approx 4 \times 10^{-4} \left( \frac{\delta m_p^*}{m_p} \right)^2 \left( \frac{m_p}{m_p^*} \right)^{1/2} \left( \frac{x_p}{0.05} \right)^{7/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6}, \quad (66)$$

and

$$\mathcal{B}' = \mathcal{B}^2. \quad (67)$$

are dimensionless parameters. These results should be compared to the parameters used by Mendell (1991b), and it is easy to confirm that the two results are in perfect agreement.

## 6 CONCLUDING REMARKS

In this paper we have derived the form of the vortex-mediated mutual friction, which arises as electrons scatter dissipatively off of the magnetic fields associated with the entrained proton currents and each neutron vortex, within the superfluid formalism developed by, for example, Prix (2004); Andersson & Comer (2005). In doing this we have made contact with previous work based on the orthodox Landau formulation (Alpar, Langer & Sauls 1984; Mendell 1991a), and demonstrated that the two pictures are consistent. Since our description incorporates the entrainment effect in a transparent way (by making the appropriate distinction between true transport velocities and momenta) this comparison lends strong support not only to our present results but to the previous work as well.



To conclude our discussion, let us put the final expression for the mutual friction force (65) to use by working out the timescale on which a difference in rotation between the two fluids in a neutron star core (eg. following a pulsar glitch) is relaxed locally. Given (8) and (65) we see that the system evolves according to

$$\left. \begin{aligned} n_n \partial_t p_i^n + \dots &= f_i^{\text{mf}} \\ n_p \partial_t p_i^p + \dots &= -f_i^{\text{mf}} \end{aligned} \right\} \longrightarrow \frac{m_p^*}{m_p} \partial_t w_i^{\text{np}} + \dots \approx -\frac{\mathcal{B} \kappa n_v}{x_p} w_i^{\text{np}}, \quad (68)$$

where, on the left-hand side we have used the definition of the momenta and assumed that the proton fraction is small (in accordance with the preceding analysis), and on the right-hand side we implicitly assume that the velocity difference is perpendicular to the vortices (that is, the two fluids rotate around the same axis). From this expression we see that the timescale on which the two fluids are dynamically coupled can be estimated by

$$\tau_d \approx \frac{m_p^*}{m_p} \frac{x_p}{\mathcal{B} \kappa n_v}. \quad (69)$$

Taking  $n_v \kappa \approx 4\pi/P$ , i.e. assuming that the two rotation rates are similar to the observed pulsar period  $P$ , cf. (16), we have

$$\tau_d \approx 10P(s) \left( \frac{m_p^*}{\delta m_p^*} \right)^2 \left( \frac{x_p}{0.05} \right)^{-1/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{-1/6}. \quad (70)$$

This estimate is about one order of magnitude smaller than the classic result of Alpar & Sauls (1988). That the two results differ is perhaps not too surprising. After all, the Alpar & Sauls (1988) analysis was based on an explicit solution for the motion of an individual vortex affected by the Magnus force (34) and the resistivity (35) given a constant relative rotation rate. In contrast, our estimate does not refer to the explicit vortex motion, only to the way that the two fluids couple via the vortices. Of course, the main astrophysical conclusion remains unaltered. The coupling timescale is much shorter than the observed relaxation timescale following (say) the large Vela glitches (Alpar & Sauls 1988). This suggests that the glitches are unlikely to be associated with the core and points instead to the superfluid neutrons in the crust playing a key role.

Basically, we have now prepared the ground for discussions of the relevance of mutual friction in different astrophysical scenarios within our formalism. This is a very important step forwards since it allows us to consider key problems concerning, for example, the mutual friction damping of pulsation modes driven unstable by gravitational radiation. As the answer may be of significance for gravitational-wave observations, the available results in that problem area (Lindblom & Mendell 1995, 2000) must be verified by independent work. Given the present analysis, we are set to carry out such calculations and expect to report the results in the not too distant future.

## ACKNOWLEDGEMENTS

NA acknowledges support from PPARC via grant no PPA/G/S/2002/00038 and Senior Research Fellowship no PP/C505791/1. GLC is supported by NSF grant no PHY-0457072.

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